Universität Münster

The cofinality of universally Baire sets problem

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The MM^{++} versus the Axiom $(*)$ problem

Asperó and Schindler have solved the MM^{++} versus the Axiom $(*)$ problem completely.

Theorem (Asperó-Schindler, 2019)

MM^{++} *implies the Axiom* $(*)$.

Definition (Woodin, 1990s)

$(*)$:

1. The Axiom of Determinacy holds in $L(\mathbb{R})$.
2. There is a $L(\mathbb{R})$-generic filter $g \subset \mathbb{P}_{\max}$ such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[g]$.

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Definition (Woodin)

$(*)^{++}$: There exists $\Gamma \subset \mathcal{P}(\mathbb{R})$ and $g \subset \mathbb{P}_{\mathrm{max}}$ such that

1. $L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+$.
2. g is $L(\Gamma, \mathbb{R})$-generic and $\mathcal{P}(\mathbb{R}) \in L(\Gamma, \mathbb{R})[g]$.

Woodin also defined $(*)^+$ and proved that $(*)^+$ is equivalent to $(*)^{++}$ in ZFC (2021).

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Theorem (Woodin)

All known models of MM^{++} *do not satisfy the Axiom* $(*)^{++}$.

Those failures are closely related to the cofinality of universally Baire sets.

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Definition (Feng-Magidor-Woodin)

A set of reals A is *universally Baire* if there are trees T and U on $\omega \times \mathrm{ON}$ such that $A = \mathrm{p}[\mathrm{T}]$ and for all posets $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \mathrm{p}[\check{\mathrm{U}}] = \mathbb{R} \setminus \mathrm{p}[\check{\mathrm{T}}].$$

Γ^{∞} denotes the set of all universally Baire sets of reals.

Assuming a proper class of Woodin cardinals, Γ^{∞} has the following nice properties:

- (Martin-Steel, Woodin) Γ^{∞} is closed under taking continuous preimages, countable joins, complements, projections.
- (Woodin) For each $A \in \Gamma^{\infty}$, $\mathrm{L}(A, \mathbb{R}) \models \mathrm{AD}^{+}$ and $\mathcal{P}(\mathbb{R}) \cap \mathrm{L}(A, \mathbb{R}) \subset \Gamma^{\infty}$.

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For an AD^+-model M,

$$\Theta^M = \sup\{\alpha \mid \exists f \in M(f\colon \mathbb{R}^M \to \alpha \text{ is surjective })\}.$$

Definition

$\theta_{uB} = \sup\{\Theta^{L(A,\mathbb{R})} \mid A \in \Gamma^\infty\}$

- Γ^∞ is prewell-ordered by Wadge reducibility $\leqslant_w$ and θ_{uB} is the length of $(\Gamma^\infty, \leqslant_w)$.
- Note that if $L(\Gamma^\infty, \mathbb{R}) \models AD^+$ and $\mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R}) = \Gamma^\infty$, then $\theta_{uB} = \Theta^{L(\Gamma^\infty,\mathbb{R})}$.
- Sealing implies that $L(\Gamma^\infty, \mathbb{R}) \models AD^+$ and $\mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R}) = \Gamma^\infty$ hold in any set generic extensions.

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- Sealing implies that $L(\Gamma^\infty, \mathbb{R}) \models AD^+$ and $\mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R}) = \Gamma^\infty$ hold in any set generic extensions.
- Under MM^{++}, $2^{\aleph_0} = \aleph_2$ and $\underset{\sim}{\delta}^1_2 = \aleph_2$. Hence $\omega_2 < \theta_{uB} \leqslant \omega_3$. Since Γ^∞ is closed under countable joins, $cf(\theta_{uB}) > \omega$. Therefore the possible values of $cf(\theta_{uB})$ are ω_1, ω_2, and ω_3.

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Assuming MM^{++} and a proper class of Woodin cardinals, $(*)^{++}$ is witnessed by Γ^∞.

Theorem (Woodin)

Assume MM^{++}. *Then* $(*)^{++}$ *implies* $\mathrm{cf}(\theta_{\mathrm{uB}}) = \theta_{\mathrm{uB}} = \omega_3$.
Moreover, assume a proper class of Woodin cardinals. Equivalent are

1. $(*)^{++}$.
2. $\mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R}) = \Gamma^\infty$ *and there exists* $L(\Gamma^\infty, \mathbb{R})$*-generic* $g \subset \mathbb{P}_{\max}$ *such that* $\mathcal{P}(\omega_2) \in L(\Gamma^\infty, \mathbb{R})[g]$.

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$(*)^{++}$ and the cofinality of universally Baire sets

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Theorem (Woodin)

Supose that κ *is supercompact and* $V[G]$ *is a* κ*-c.c. extension in which* $\kappa = \omega_2$. *Then in* $V[G]$ $\mathrm{cf}(\theta_{\mathrm{uB}})$ *is either* ω_1 *or* ω_2. *Hence in the standard models of* MM^{++}, $(*)^{++}$ *fails.*

But the exact value of $\mathrm{cf}(\theta_{\mathrm{uB}})$ is unknown in the standard models of MM^{++}.

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Which values of $\mathrm{cf}(\theta_{\mathrm{uB}})$ are compatible with MM^{++} (or fragments of MM^{++})?

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Fistly, Sealing itself seems nothing to do with the value of $\mathrm{cf}(\theta_{\mathrm{uB}})$.

Theorem (Blue-Sargsyan)

For each $i \in \{1, 2, 3\}$, *Sealing*$+ \mathrm{cf}(\theta_{\mathrm{uB}}) = \omega_i$ *is consistent.*

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Theorem (Blue-Sargsyan)

For each $i \in \{1, 2, 3\}$, *Sealing* $+ \mathrm{cf}(\theta_{uB}) = \omega_i$ *is consistent.*

$\mathbb{P}_{max}$ forcing is the only known way to obtain models of $(*)^{++}$ (and $\mathrm{cf}(\theta_{uB}) = \omega_3$).

Theorem

- *(Woodin)* $\mathrm{ZFC} + \mathrm{MM}^{++}(\mathfrak{c}) + (*)^{++}$ *is consistent.*
- *(Schindler-Y., 2024)* $\mathrm{ZFC} + \mathrm{MM}^{*,++}_{\mathfrak{c}} + (*)^{++}$ *is consistent.*
- *(Blue-Larson-Sargsyan, 2025) For each* $n \in [3, \omega)$, $\mathrm{ZFC} + \mathrm{MM}^{++}(\mathfrak{c}) + \forall i \in [2, n] \neg\Box(\aleph_i) + (*)^{++}$ *is consistent.*

We know that $\mathrm{cf}(\theta_{\mathrm{uB}})$ is eihter ω_1 or ω_2 in the standard models of MM^{++}.
To compute the exact value, it is necessary to understand how SSP forcings can change the structure of $(\Gamma^\infty, \leqslant_w)$.

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Question

When and which kind of SSP forcing $\mathbb{P}$ adds a new universally Baire sets $\dot{B}$ such that for all $A \in \Gamma^\infty$, $\Vdash_{\mathbb{P}} A^{\dot{G}} <_w \dot{B}$?

If $\mathbb{P}$ does not add such a new universally Baire set, we say $\mathbb{P}$ is *uB-bounding*.

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- (Woodin) If NS_{ω_1} is saturated and $\mathcal{P}(\omega_1)^\#$ exists, then $\underset{\sim}{\delta}^1_2 = \aleph_2$.
- If $\mathrm{cf}(\theta_{\mathrm{uB}}) > \omega_1$, then Namba forcing is not uB-bounding.
- (Woodin) Under $\mathrm{MM}^{++} + \mathrm{cf}(\theta_{\mathrm{uB}}) = \omega_1$+(weak) UBH, every SSP forcing is uB-bounding.

Proper forcings and the cofinality of θ_{uB}

Theorem (Neeman-Zapletal)

Let δ be a weakly compact Woodin cardinal and $\mathbb{P}$ be a proper forcing of size $< \delta$. Then in $V^{\mathbb{P}}$ there is an elementary embedding $j \colon L(\mathbb{R})^V \to L(\mathbb{R})^{V^{\mathbb{P}}}$ which fixes all ordinals. Hence $\mathbb{P}$ does not change $\Theta^{L(\mathbb{R})}$.

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Foreman-Magidor showed more general results for reasonable forcings.

Theorem

Assume Sealing and a proper class of Woodin cardinals. Let δ be good Woodin limit of Woodins and $\mathbb{P}$ be a proper forcing of size $< \delta$. Then $\Vdash_{\mathbb{P}} \theta_{\mathrm{uB}}^{\check{V}} = \theta_{\mathrm{uB}}$. Moreover, in $V^{\mathbb{P}}$ there is an elementary embedding $j\colon L(\Gamma^\infty, \mathbb{R})^V \to L(\Gamma^\infty, \mathbb{R})^{V^{\mathbb{P}}}$ which fixes all ordinals.

We say a Woodin cardinal δ is *good* if whenever $G \subset \mathbb{Q}_{<\delta}$ is V-generic, then $j_G(\Gamma^\infty) = (\Gamma^\infty)^{V[G]}$.

Theorem

$\mathsf{PFA} + \mathrm{cf}(\theta_{\mathrm{uB}}) = \omega_1$ *is consistent.*

Questions

There are many open questions around the cofinality of θ_{uB} problem. I list some of them:

1. Assume $\mathrm{cf}(\theta_{\mathrm{uB}}) = \omega_1$. Is every SSP forcing uB-bounding?
2. What are the possible values of $\mathrm{cf}(\theta_{\mathrm{uB}})$ in Namba forcing extensions?
3. Is the following statement compatible with MM^{++}?
   "There exists $A \subset \mathbb{R}$ such that $(A, \mathbb{R}) \models \mathsf{AD}^+$ and Γ^∞ is contained in the Suslin-co-Suslin sets of $L(A, \mathbb{R})$"

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Thank you for listening!!

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